Bloch–Wilson Hamiltonian and a Generalization of the Gell-Mann–Low Theorem[1]

Axel Weber[2]

Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C–3 Cd. Universitaria, A. P. 2–82, 58040 Morelia, Michoacán, Mexico

Abstract. The effective Hamiltonian introduced many years ago by Bloch and generalized later by Wilson, appears to be the ideal starting point for Hamiltonian perturbation theory in quantum field theory. The present contribution derives the Bloch–Wilson Hamiltonian from a generalization of the Gell-Mann–Low theorem, thereby enabling a diagrammatic analysis of Hamiltonian perturbation theory in this approach.

The presently available techniques for calculations in quantum field theory reflect the dominance of scattering processes for the experimental exploration of the physics of elementary particles. The single most important technique is beyond doubt Lagrangian perturbation theory, the explicit covariance of which has historically played an important rôle in the implementation of the renormalization program. This in turn was the crucial ingredient for converting the formal expressions of Lagrangian perturbation theory into predictions for measurable quantities. On the other hand, the identification of physical states defined as eigenstates of the Hamiltonian and the Hilbert space they span, becomes a complicated task in this approach, which is exemplified by the serious problems arising in the solution of the Bethe–Salpeter equation. In short, Lagrangian perturbation theory is primarily a theory of processes as opposed to a theory of states.

This contribution is concerned with the development of a theory of states, establishing efficient techniques for Hamiltonian perturbation theory. Apart from the possibility of gaining a new perspective on the foundations of quantum field theory, this approach appears to be natural for the description of hadronic structure and of bound state phenomena in general. In a very general

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[2)] e–mail: axel@io.ifm.umich.mx

setting, consider the problem of solving the Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle \tag{1}$$

for the state $|\psi\rangle$. The Hamiltonian is supposed to be decomposable into a "free" and an "interacting" part, $H = H_0 + H_I$, where the eigenstates of H_0 are explicitly known and span the full Hilbert space (or Fock space) $\mathcal{F}$, which we picture as a direct sum of free n–particle subspaces ($n \geq 0$). The eigenstates of H are expected to be representable as (infinite) linear combinations of the eigenstates of H_0, hence the Schrödinger equation (1) can be written in a Fock space basis, where in general an infinite number of n–particle subspaces are involved. The problem in this generality is obviously too difficult to be solved in practice.

Restricting attention momentarily to the vacuum state, the Gell-Mann–Low theorem [1] states that the free (Fock space) vacuum evolves dynamically into the physical vacuum as H_0 turns adiabatically into H. Explicit expressions can then be given for the physical vacuum state and its energy in terms of the free n–particle states and their energies in the form of a perturbative series. It is natural to ask whether it is possible to generalize the theorem to the case where the perturbative vacuum is replaced by a linear subspace Ω of $\mathcal{F}$ consisting of eigenspaces of H_0, i.e. $H_0\Omega \subseteq \Omega$, the simplest non–trivial example being the free two–particle subspace of $\mathcal{F}$. When the interaction H_I is switched on adiabatically, one may expect that Ω evolves into the suspace of interacting physical two–particle states, where now different eigenstates of H_0 are allowed to mix during the adiabatic process. If this expectation comes true, the determination of the physical two–particle states may be reduced to a problem within the free two–particle subspace, thus dramatically reducing the number of degrees of freedom to be considered and converting the problem into a (at least numerically) solvable one.

Couched into mathematical jargon, what one is looking for is a map U_{BW} from Ω to a direct sum of eigenspaces of H, i.e. $HU_{BW}\Omega \subseteq U_{BW}\Omega$, where U_{BW} is expected to be related to the adiabatic evolution operator. One would then hope that U_{BW} induces a similarity transformation, so that the problem of diagonalizing H in $U_{BW}\Omega$ is equivalent to diagonalizing $H_{BW} := U_{BW}^{-1} H U_{BW} : \Omega \to \Omega$, which in the example above is equivalent to a relativistic two–particle Schrödinger equation. The simplest (but not unique) choice for $U_{BW}^{-1} : U_{BW}\Omega \to \Omega$ is the orthogonal projector P to Ω,[3] hence we will look for an operator U_{BW} in Ω with

$$PU_{BW} = P = \mathbf{1}_\Omega \, . \tag{2}$$

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[3] That the choice of P for the similarity transformation is not unreasonably simple is suggested by phenomenology: even in the highly non–perturbative situation of low–energy QCD the physical hadrons can be associated with a specific content of constituent quarks (and thus with an element of the free two– or three–particle subspace).

Eq. (2) implies in turn the injectivity of U_{BW}, hence also $U_{BW}P = \mathbf{1}$ in $U_{BW}\Omega$. Together with $HU_{BW}\Omega \subseteq U_{BW}\Omega$ one then has that

$$(\mathbf{1} - U_{BW}P)HU_{BW} = 0 \text{ in } \Omega\,. \tag{3}$$

Eqs. (2) and (3) together in fact characterize U_{BW}: (3) implies $HU_{BW}\Omega = U_{BW}(PHU_{BW}\Omega) \subseteq U_{BW}\Omega$. Consequently, $H|_{U_{BW}\Omega}$ is diagonalizable, and by (2) it is a similarity transform of H_{BW}.

Remarkably, Eqs. (2) and (3) also determine U_{BW} uniquely, at least within the perturbative regime. To see this, rewrite (3) as

$$\begin{aligned} H_I U_{BW} - U_{BW}PH_I U_{BW} &= U_{BW}PH_0U_{BW} - H_0U_{BW} \\ &= U_{BW}H_0 - H_0U_{BW}\,, \end{aligned} \tag{4}$$

where I have used $PH_0U_{BW} = H_0PU_{BW} = H_0$. Now consider the matrix element of (4) between $\langle u|$ and $|k\rangle$, where $|k\rangle \in \Omega$ and $|u\rangle \in \Omega^\perp$ (the orthogonal complement of Ω in $\mathcal{F}$) are eigenstates of H_0 with eigenvalues E_k and E_u, respectively,

$$\langle u|H_I U_{BW} - U_{BW}PH_I U_{BW}|k\rangle = (E_k - E_u)\langle u|U_{BW}|k\rangle\,. \tag{5}$$

It then follows that

$$\begin{aligned} U_{BW} &= P + (\mathbf{1} - P)U_{BW}P \\ &= P + \int_\Omega dk \int_{\Omega^\perp} du\, |u\rangle\langle u|U_{BW}|k\rangle\langle k| \\ &= P + \int_\Omega dk \int_{\Omega^\perp} du\, |u\rangle \frac{\langle u|H_I U_{BW} - U_{BW}PH_I U_{BW}|k\rangle}{E_k - E_u}\langle k|\,, \end{aligned} \tag{6}$$

where I have taken k and u to label the eigenstates of H_0 in Ω and $\Omega^\perp$, respectively. Eq. (6) can be solved iteratively to obtain U_{BW} as a power series in H_I. It should be emphasized, however, that the individual terms in the series are not guaranteed to give convergent expressions (let alone the series as a whole). This depends, among other things, on the choice of Ω.

Eqs. (2) and (3) have been used for the characterization of U_{BW} before, first by Bloch [2] in the context of degenerate quantum mechanical perturbation theory, and later by Wilson [3] for the formulation of a non–perturbative renormalization group in Minkowski space. In practical applications, one will calculate U_{BW} to a certain order in the iterative expansion of (6) and solve the Schrödinger equation for the corresponding Hamiltonian $H_{BW} = PHU_{BW}$. Its solution yields an approximation to the eigenvalues of $H|_{U_{BW}\Omega}$ (the eigenvalues are invariant under similarity transformations) and also to the eigenstates via $|\psi\rangle = U_{BW}|\phi\rangle$ where $|\phi\rangle$ are the eigenstates of H_{BW}. The solutions will in general also include bound states (e.g., if Ω is the free two–particle subspace), in contrast to Lagrangian perturbation theory. The reason for this difference

is that although in the present formalism H_{BW} is determined perturbatively, the corresponding Schrödinger equation can be solved exactly (at least to arbitrary precision with numerical methods). This is somewhat analogous to the Bethe–Salpeter equation, but avoids the conceptual problems associated with the latter. In this context, it is worth mentioning that the normalizability of the free two–particle component $|\phi\rangle = P|\psi\rangle$ gives a natural criterium for the "boundedness" of the state $|\psi\rangle$, although the latter may not be normalizable in the Hilbert space sense.

The formulation presented so far has two important shortcomings: first, the terms in the perturbative series following from (6) are not well–defined in the case of vanishing energy denominators, and a consistent prescription is at least not obvious from (3) or (6). Second, it is not a priori clear how to translate the terms in the perturbative series into diagrams. A diagrammatic formulation, however, is expected to be at least helpful, if not imperative, for the investigation of such important properties as renormalizability and Lorentz and gauge invariance at finite orders of the expansion, as well as for practical applications of the formalism.

In search of an alternative characterization of U_{BW}, I will now return to the idea of the adiabatic evolution. Consider the adiabatic evolution operator from $t = -\infty$ to $t = 0$,

$$\begin{aligned} U_\epsilon &= T \exp -i \int_{-\infty}^{0} dt\, e^{-\epsilon|t|} H_I(t) \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{0} dt_1 \cdots \int_{-\infty}^{0} dt_n\, e^{-\epsilon(|t_1|+\ldots+|t_n|)}\, T[H_I(t_1)\cdots H_I(t_n)]\,, \end{aligned} \tag{7}$$

where

$$H_I(t) = e^{iH_0 t} H_I\, e^{-iH_0 t} \tag{8}$$

is the usual expression in the interaction picture and T stands for the decreasing time ordering operator. Then the following theorem holds:

Generalized Gell-Mann–Low Theorem. *With the notations introduced before, suppose that the operator* $U_{BW} := \lim_{\epsilon\to 0} U_\epsilon (PU_\epsilon P)^{-1}$ *exists in* Ω. *Then it has the properties* $PU_{BW} = P$ *and* $(\mathbf{1} - U_{BW}P)HU_{BW} = 0$ *in* Ω.

Remarks. We thus have an explicit expression for U_{BW} in terms of the adiabatic evolution operator. Given that $PU_\epsilon P$ is always formally invertible as a power series in H_I, the implications of the theorem rest on the existence of the limit $\epsilon \to 0$ of $U_\epsilon (PU_\epsilon P)^{-1}$, which in turn depends on the choice of Ω.

Proof. The property $PU_{BW} = P$ follows directly from the definition of U_{BW}. The first part of the proof of $(\mathbf{1} - U_{BW}P)HU_{BW} = 0$ is identical to the original Gell-Mann–Low proof [1] and will not be reproduced here. It establishes by manipulation of the series (7) for U_ϵ that (before taking the limit $\epsilon \to 0$)

$$HU_\epsilon = U_\epsilon H_0 + i\epsilon g \frac{\partial}{\partial g} U_\epsilon \,, \tag{9}$$

where H_I is assumed to be proportional to some "coupling constant" g.

Now choose any $|\phi\rangle \in \Omega$. Eq. (9) implies

$$HU_\epsilon (PU_\epsilon P)^{-1}|\phi\rangle = U_\epsilon H_0 (PU_\epsilon P)^{-1}|\phi\rangle + i\epsilon \left(g \frac{\partial}{\partial g} U_\epsilon \right) (PU_\epsilon P)^{-1}|\phi\rangle \,. \tag{10}$$

It follows that

$$\begin{aligned} &HU_\epsilon (PU_\epsilon P)^{-1}|\phi\rangle - i\epsilon g \frac{\partial}{\partial g} \left(U_\epsilon (PU_\epsilon P)^{-1} \right) |\phi\rangle \\ &\quad = U_\epsilon H_0 (PU_\epsilon P)^{-1}|\phi\rangle + i\epsilon U_\epsilon (PU_\epsilon P)^{-1} \left(Pg \frac{\partial}{\partial g} U_\epsilon \right) (PU_\epsilon P)^{-1}|\phi\rangle \qquad (11) \\ &\quad = U_\epsilon (PU_\epsilon P)^{-1} P H U_\epsilon (PU_\epsilon P)^{-1}|\phi\rangle \,, \qquad (12) \end{aligned}$$

where in going from (11) to (12) Eq. (10) has been used again, multiplied by $U_\epsilon (PU_\epsilon P)^{-1} P$ from the left, and P has been inserted to the left of H_0, which is possible due to $H_0 \Omega \subseteq \Omega$. Taking the limit $\epsilon \to 0$, we have $HU_{BW}|\phi\rangle = U_{BW} P H U_{BW}|\phi\rangle$, which proves the theorem. In taking the limit, the existence of the g–derivative of U_{BW} in Ω has been assumed. Incidentally, this assumption implies that the expression $U_\epsilon (g\,\partial/\partial g)(PU_\epsilon P)^{-1}|\phi\rangle$ is in general *divergent* in the limit $\epsilon \to 0$, since $HU_\epsilon (PU_\epsilon P)^{-1}|\phi\rangle$ cannot be expected to be equal to $U_\epsilon H_0 (PU_\epsilon P)^{-1}|\phi\rangle$ in this limit [1]. ■

The theorem corroborates the expectation detailed at the beginning of this contribution. More importantly, the adiabatic formulation also has the benefit of fixing an $i\epsilon$–prescription for the energy denominators appearing in the series generated by (6). Performing the time integrations in $U_\epsilon (PU_\epsilon P)^{-1}$ yields explicitly to second order in H_I

$$\begin{aligned} U_{BW} = &\int_\Omega dk\, |k\rangle\langle k| + \int_\Omega dk \int_{\Omega^\perp} du\, |u\rangle \frac{\langle u|H_I|k\rangle}{E_k - E_u + i\epsilon} \langle k| \\ &- \int_\Omega dk\, dk' \int_{\Omega^\perp} du\, |u\rangle \frac{\langle u|H_I|k'\rangle\, \langle k'|H_I|k\rangle}{(E_k - E_u + 2i\epsilon)(E_{k'} - E_u + i\epsilon)} \langle k| \\ &+ \int_\Omega dk \int_{\Omega^\perp} du\, du'\, |u\rangle \frac{\langle u|H_I|u'\rangle\, \langle u'|H_I|k\rangle}{(E_k - E_u + 2i\epsilon)(E_k - E_{u'} + i\epsilon)} \langle k| + \dots \,, \end{aligned} \tag{13}$$

where the limit $\epsilon \to 0$ is understood. The same expression without the $i\epsilon$–prescription follows from iterating (6).

The second important advantage of the formulation in terms of U_ϵ is the ready translation into diagrams. The diagrams associated with the perturbative expansion of H_{BW} turn out to be similar to Goldstone or time–ordered diagrams, but unlike the latter they do *not* combine into a set of Feynman diagrams. This is essentially due to the fact that the matrix elements of the effective Hamiltonian $\langle k|HU_{BW}|k'\rangle$ in general do not vanish if the energies E_k and $E_{k'}$ are different.

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